

# Fine structure of spectrum of twist-three operators in QCD.

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## Abstract

We unravel the structure of the spectrum of the anomalous dimensions of the quark-gluon twist-3 operators which are responsible for the multiparton correlations in hadrons and enter as a leading contribution to several physical cross sections. The method of analysis is based on the recent finding of a non-trivial integral of motion for the corresponding Hamiltonian problem in multicolour limit which results into exact integrability of the three-particle system. Quasiclassical expansion is used for solving the problem. We address the chiral-odd sector as a case of study.

Keywords: twist-three operators, evolution, three-particle problem, integrability, spectrum of eigenvalues

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# 1 Introduction.

The studies of power suppressed (in a hard momentum scale) phenomena in physical cross sections give an important insight into underlying physics of strong interaction since corresponding quantities manifest multiparton correlations in hadrons and can shed some light on the complicated dynamics of the hadronic substructure. The most promising is the study of twist-three contributions in hadronic reactions which enter as a leading effect in certain asymmetries. The planned and running experiments will allow to extract these quantities and to face them with theoretical predictions. However, the question arises about the scale at which experimental and theoretical predictions are confronted. This requires the knowledge of the evolution equations which govern the scale dependence of the corresponding correlation functions — a generalization of the familiar one-variable parton densities — as well as their solutions, i.e. determination of their eigenvalues and eigenstates. While the former problem is quite straightforward since it can be easily resolved using standard methods of QCD perturbation theory, provided a basis of operators mixed under renormalization is chosen in an appropriate way, and this problem has been tackled by a number of authors (see e.g. reviews [1, 2]). On the other hand the second problem is by far more complicated since it requires the solution of the Faddeev-type many-body equations for the particles with pair-wise interaction. Recently a breakthrough has been made, however, by the authors of Ref. [3] where they have found an additional non-trivial integral of motion for the problem at hand. Even so, when the problem acquires an equivalence to an exactly solvable model the task is proved to be too complicated to be solved exactly analytically. Therefore, it is welcome to develop an approximate scheme which could allow for a systematical improvement of approximations involved. The present paper is devoted to the study along this line for the chiral-odd twist-three evolution equations, i.e. the ones related to the hadron structure functions  $h_L(x)$  and  $e(x)$  [4].

## 2 Quasi-partonic operators and their evolution.

The twist-3 sector is exceptional as compared to even higher twists since in the former case contrary to the latter we can reduce the renormalization group analysis to the study of the UV divergencies of the so-called quasi-partonic operators [5] which form a complete basis of functions. The twist of these objects equals the number of fields which the composite operators are constructed from. This means that due to the fact that in leading order only a pair-wise interaction of partons is of relevance, the corresponding kernels in sub-channels are the familiar twist-2 non-forward evolution kernels.

The generic form of the evolution equation for twist-3 correlation functions in the momentum

fraction formalism looks like

$$\mu^2 \frac{d}{d\mu^2} Z(x_1, x_2, x_3) = \int \prod_{i=1}^3 dx'_i \delta \left( \sum_{i=1}^3 x'_i - \zeta \right) \mathbf{K}(\{x_i\}|\{x'_i\}) Z(x'_1, x'_2, x'_3), \quad (1)$$

with the constraint  $x_1 + x_2 + x_3 = \zeta$  for the momentum fractions of partons imposed on the both sides of this equation. The variable  $\zeta$  stands for the  $+$ -component of the  $t$ -channel momentum: the limit  $\zeta = 0$  corresponds to the usual DIS kinematics (and Eq. (1) to a generalized DGLAP equation), and  $\zeta = 1$  to the exclusive one (Brodsky-Lepage equation for baryon distribution amplitude). The kernel  $\mathbf{K}$  in leading order has a pair-wise structure:

$$\mathbf{K}(\{x_i\}|\{x'_i\}) = \sum_{i < j} K(x_i, x_j | x'_i, x'_j) \delta(x_i + x_j - x'_i - x'_j), \quad (2)$$

with  $K(x_i, x_j | x'_i, x'_j)$  being the interaction kernel of two nearby particles.

For the purposes of the present study we need the quark-anti-quark and (anti-)quark-gluon kernels with non-contracted Lorentz, Dirac and colour indices. They can be decomposed into independent structures as follows

$$\begin{aligned} & {}^{q\bar{q}}K_{i'j';\alpha'\beta'}^{ij;\alpha\beta}(x_1, x_2 | x'_1, x'_2) \\ &= -\frac{\alpha_s}{2\pi} \left\{ \frac{1}{4} [(\gamma_-)_{\alpha\beta} (\gamma_+)_{\beta'\alpha'} - (\gamma_- \gamma_5)_{\alpha\beta} (\gamma_+ \gamma_5)_{\beta'\alpha'}] \left[ \frac{1}{C_A} \delta_{ij} \delta_{j'i'} {}^{q\bar{q}}K_{(1)}^V + \frac{1}{T_F} (t^a)_{ij} (t^a)_{j'i'} {}^{q\bar{q}}K_{(8)}^V \right] \right. \\ & \quad \left. + \frac{1}{4} (\sigma_{-\mu}^\perp)_{\alpha\beta} (\sigma_{+\mu}^\perp)_{\beta'\alpha'} \left[ \frac{1}{C_A} \delta_{ij} \delta_{j'i'} {}^{q\bar{q}}K_{(1)}^T + \frac{1}{T_F} (t^a)_{ij} (t^a)_{j'i'} {}^{q\bar{q}}K_{(8)}^T \right] \right\} (x_1, x_2 | x'_1, x'_2), \quad (3) \end{aligned}$$

for quark-anti-quark kernel, and

$$\begin{aligned} & {}^{qg}K_{i'a';\alpha'\mu'}^{ia;\alpha\mu}(x_1, x_2 | x'_1, x'_2) \\ &= -\frac{\alpha_s}{2\pi} \frac{1}{C_F C_A} (t^a)_{ij} (t^a)_{j'i'} \left\{ -\frac{1}{8} \left[ (\gamma_- \gamma_\mu^\perp)_{\alpha\beta} (\gamma_+ \gamma_\mu^\perp)_{\beta'\alpha'} + (\gamma_- \gamma_\mu^\perp \gamma_5)_{\alpha\beta} (\gamma_+ \gamma_\mu^\perp \gamma_5)_{\beta'\alpha'} \right] {}^{qg}K_{(3)}^V \right. \\ & \quad \left. - \frac{1}{4} (\tau_{\mu\nu;\rho\sigma}^\perp \gamma_- \gamma_\nu^\perp)_{\alpha\beta} (\tau_{\mu'\nu';\rho\sigma}^\perp \gamma_+ \gamma_{\nu'}^\perp)_{\beta'\alpha'} {}^{qg}K_{(3)}^T \right\} (x_1, x_2 | x'_1, x'_2) \\ & + \dots, \quad (4) \end{aligned}$$

for the quark-gluon one. The ellipsis stand for other colour structures which, however, are irrelevant for our present consideration. Here as usual the  $\pm$ -subscripts mean the projection on two opposite tangents to the light cone, and the transversity tensor  $\tau_{\mu\nu;\rho\sigma}^\perp \equiv \frac{1}{2} (g_{\mu\rho}^\perp g_{\nu\sigma}^\perp + g_{\mu\sigma}^\perp g_{\nu\rho}^\perp - g_{\mu\nu}^\perp g_{\rho\sigma}^\perp)$  is constructed from the 2D metric tensors of the transverse plane  $g_{\mu\nu}^\perp = g_{\mu\nu} - n_\mu n_\nu^* - n_\nu n_\mu^*$ .

From diagrams in Fig. 1 we easily get<sup>2</sup> (cf. [5, 6, 7])

$$\frac{1}{C_F} {}^{q\bar{q}}K_{(1)}^V(x_1, x_2 | x'_1, x'_2) = \frac{1}{C_F - \frac{C_A}{2}} {}^{q\bar{q}}K_{(8)}^V(x_1, x_2 | x'_1, x'_2)$$

<sup>2</sup>Here  $\Theta_{i_1 i_2 \dots i_n}^m(x_1, x_2, \dots, x_n) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi i} \alpha^m \prod_{k=1}^n (\alpha x_k - 1 + i0)^{-i_k}$ . A detailed discussion of the properties of these functions can be found in Ref. [2].

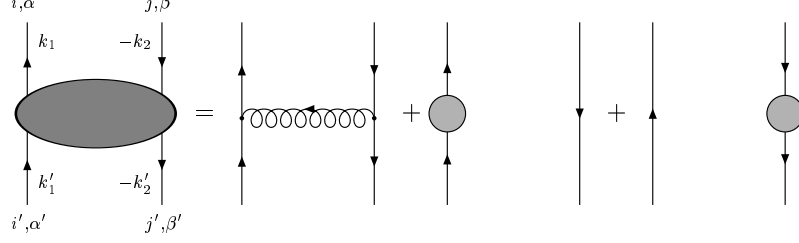


Figure 1: The diagrams contributing to the pair-wise quark-anti-quark kernel at one-loop order in the light-cone gauge,  $B_+ = 0$ . The blobs on the lines stand for the wave function renormalization counterterm.

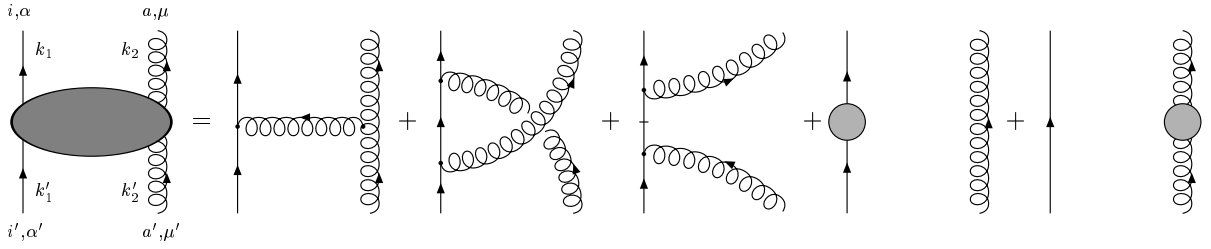


Figure 2: Same as in Fig. 1 but for the quark-gluon kernel. The graph with the crossed fermion propagator corresponds to the contact-type contribution arising from the use of the Heisenberg equation of motion for the quark field.

$$\begin{aligned}
&= \left[ \frac{x_1}{x_1 - x'_1} \Theta_{11}^0(x_1, x_1 - x'_1) + \frac{x_2}{x_2 - x'_2} \Theta_{11}^0(x_2, x_2 - x'_2) + \Theta_{111}^0(x_1, -x_2, x_1 - x'_1) \right]_+, \quad (5) \\
\frac{1}{C_F} {}^{q\bar{q}}K_{(1)}^T(x_1, x_2|x'_1, x'_2) &= \frac{1}{C_F - \frac{C_A}{2}} {}^{q\bar{q}}K_{(8)}^T(x_1, x_2|x'_1, x'_2) \\
&= \left[ \frac{x_1}{x_1 - x'_1} \Theta_{11}^0(x_1, x_1 - x'_1) + \frac{x_2}{x_2 - x'_2} \Theta_{11}^0(x_2, x_2 - x'_2) \right]_+ + \frac{1}{2} \delta(x_1 - x'_1), \quad (6)
\end{aligned}$$

and from Fig. 2 we have (cf. [5])

$$\begin{aligned}
&{}^{qg}K_{(3)}^V(x_1, x_2|x'_1, x'_2) \\
&= \frac{C_A}{2} \left[ \frac{x_1}{x_1 - x'_1} \Theta_{11}^0(x_1, x_1 - x'_1) \right]_+ + \frac{C_A}{2} \frac{x_2}{x'_2} \left[ \frac{x_2}{x_2 - x'_2} \Theta_{11}^0(x_2, x_2 - x'_2) \right]_+ \\
&+ \frac{C_A}{2} \frac{x_2 + x'_2}{x'_2} \Theta_{111}^0(x_1, -x_2, x_1 - x'_1) - \left( C_F - \frac{C_A}{2} \right) \frac{x_2 - x'_1}{x'_2} \Theta_{111}^0(x_1, -x_2, x_1 - x'_2) \\
&+ C_F \frac{x_2}{x'_2} \frac{x_1}{x_1 + x_2} \Theta_{11}^0(x_1, -x_2) + \frac{1}{8} (\beta_0 + 5C_A) \delta(x_1 - x'_1), \quad (7) \\
&{}^{qg}K_{(3)}^T(x_1, x_2|x'_1, x'_2) \\
&= \frac{C_A}{2} \left[ \frac{x_1}{x_1 - x'_1} \Theta_{11}^0(x_1, x_1 - x'_1) \right]_+ + \frac{C_A}{2} \frac{x_2}{x'_2} \left[ \frac{x_2}{x_2 - x'_2} \Theta_{11}^0(x_2, x_2 - x'_2) \right]_+
\end{aligned}$$

$$-\left(C_F - \frac{C_A}{2}\right) \frac{x_1}{x'_2} \Theta_{11}^0(x_1, x_1 - x'_2) + \frac{1}{8}(\beta_0 + 5C_A)\delta(x_1 - x'_1). \quad (8)$$

With these results at hand the construction of the evolution equation for the quasi-partonic operators is almost trivial since it requires a mere evaluation of simple traces of Eqs. (3,4) with tensor structure of the composite operator under study.

The correlators related to the chiral-odd structure function  $e(x)$  and  $h_L(x)$  are given by light-cone Fourier transformation [8]

$$Z(x_1, x_2 = -x_1 - x_3, x_3) = \int \frac{d\kappa_1}{2\pi} \frac{d\kappa_3}{2\pi} e^{i\kappa_1 x_1 + i\kappa_3 x_3} \langle h | \frac{1}{2} \{ \mathcal{Z}(\kappa_1, 0, \kappa_3) \pm \mathcal{Z}(-\kappa_3, 0, -\kappa_1) \} | h \rangle \quad (9)$$

of the non-local operators

$$\mathcal{Z}(\kappa_1, \kappa_2, \kappa_3) = \frac{1}{2} \bar{\psi}(\kappa_3 n) \sigma_{\mu+}^\perp \begin{pmatrix} 1 \\ \gamma_5 \end{pmatrix} t^a g G_{+\mu}^a(\kappa_2 n) \psi(\kappa_1 n). \quad (10)$$

The “+” and “−” signs in Eq. (9) correspond to 1 and  $\gamma_5$  structures and, i.e. to  $e$  and  $h_L$  functions, respectively. Thus, we have finally the evolution kernel which governs the scale dependence of the correlator  $Z(x_1, x_2, x_3)$

$$\begin{aligned} \mathbf{K}^{\text{odd}}(\{x_i\}|\{x'_i\}) = & -\frac{\alpha_s}{2\pi} \left\{ {}^{q\bar{q}}K_{(8)}^T(x_1, x_3|x'_1, x'_3) \delta(x_1 + x_3 - x'_1 - x'_3) \right. \\ & + {}^{qg}K_{(3)}^V(x_1, x_2|x'_1, x'_2) \delta(x_1 + x_2 - x'_1 - x'_2) \\ & \left. + {}^{qg}K_{(3)}^V(x_3, x_2|x'_3, x'_2) \delta(x_3 + x_2 - x'_3 - x'_2) - \frac{\beta_0}{4} \delta(x_1 - x'_1) \delta(x_2 - x'_2) \right\}, \end{aligned} \quad (11)$$

where we have added the charge renormalization piece due to presence of the coupling constant in the definition of the composite operator (10). Using explicit expressions for pair-wise kernels we can find that this is exactly the result of Ref. [8] obtained by a different method.

### 3 Consequences of conformal invariance.

It is well-known that the tree-level conformal invariance of the theory is enough for diagonality of the one-loop anomalous dimensions matrix of the conformal operators [9]-[13]. This means that the pair-wise kernels displayed above can be diagonalized in the basis of the Jacobi polynomials,  $P_j^{\{1,2\},1}$ . Namely, due to the support properties of the kernels we can write

$$\begin{aligned} \int dx_1 dx_2 \delta(x_1 + x_2 - x'_1 - x'_2) P_j^{\{1,2\},1} \left( \frac{x_1 - x_2}{x_1 + x_2} \right) {}^{q\{\bar{q},g\}}K^\Gamma(x_1, x_2|x'_1, x'_2) \\ = \frac{1}{2} {}^{q\{\bar{q},g\}}\gamma_j^\Gamma P_j^{\{1,2\},1} \left( \frac{x'_1 - x'_2}{x'_1 + x'_2} \right), \end{aligned} \quad (12)$$

where  $j$  is a conserved (in leading order) quantum number related to the conformal spin of the composite two-particle operators, i.e. the eigenvalue of the Casimir operator of the collinear conformal algebra  $su(1, 1)$ . The eigenvalues of the kernels are<sup>3</sup>

$${}^{q\bar{q}}\gamma_{(8)j}^V = \left(C_F - \frac{C_A}{2}\right) (2\psi(j+1) + 2\psi(j+3) - 4\psi(1) - 3), \quad (13)$$

$${}^{q\bar{q}}\gamma_{(8)j}^T = \left(C_F - \frac{C_A}{2}\right) (4\psi(j+2) - 4\psi(1) - 3), \quad (14)$$

$${}^{qg}\gamma_{(3)j}^V = \frac{C_A}{2} (2\psi(j+1) + 2\psi(j+4) - 4\psi(1)) + \left(C_F - \frac{C_A}{2}\right) \frac{4\sigma(j)}{(j+1)(j+2)(j+3)} + \frac{1}{4} (\beta_0 - 3C_A), \quad (15)$$

$${}^{qg}\gamma_{(3)j}^T = \frac{C_A}{2} (2\psi(j+2) + 2\psi(j+3) - 4\psi(1)) - \left(C_F - \frac{C_A}{2}\right) \frac{2\sigma(j)}{j+2} + \frac{1}{4} (\beta_0 - 3C_A). \quad (16)$$

where  $\sigma(j) = (-1)^j$ . Using this we can deduce the following representation of the kernels [7]

$$\phi_1\phi_2 K(x_1, x_2 | x'_1, x'_2) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{w(x_1, \nu_1 | x_2, \nu_2)}{\omega_j(\nu_1, \nu_2)} P_j^{(\nu_2, \nu_1)} \left( \frac{x_1 - x_2}{x_1 + x_2} \right) \phi_1\phi_2 \gamma_j P_j^{(\nu_2, \nu_1)} \left( \frac{x'_1 - x'_2}{x'_1 + x'_2} \right), \quad (17)$$

where  $w(x_1, \nu_1 | x_2, \nu_2) = x_1^{\nu_1} x_2^{\nu_2}$  and  $\omega_j(\nu_1, \nu_2) = \frac{\Gamma(j+\nu_1+1)\Gamma(j+\nu_2+1)}{(2j+\nu_1+\nu_2+1)j!\Gamma(j+\nu_1+\nu_2+1)}$ , with  $\nu_\ell = d_\ell + s_\ell - 1$ , and  $d_\ell$  and  $s_\ell$  being canonical scale dimension and spin of the constituent  $\phi_\ell = \{q, g\}$ .

In the basis of local operators the above eigenfunctions correspond to the conformal operators [11, 12]

$$\mathcal{O}_{jl} = \phi_2 (i\partial_+)^l P_j^{(\nu_2, \nu_1)} \left( \overleftrightarrow{\partial}_+ / \partial_+ \right) \phi_1, \quad (18)$$

where  $\phi_\ell$  is an arbitrary local field. These operators form an irreducible representation in the space of bilinear composite operators of the collinear conformal group with generators  $\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}^3$ . Here the step-up  $\mathcal{J}^+ = i\mathcal{P}_+$ , the step-down  $\mathcal{J}^- = \frac{i}{2}\mathcal{K}_-$  and the grade  $\mathcal{J}^3 = \frac{i}{2}(\mathcal{D} + \mathcal{M}_{-+})$  operators, constructed from the momentum,  $\mathcal{P}_+$ , the angular momentum,  $\mathcal{M}_{-+}$ , the dilatation,  $\mathcal{D}$ , and the special conformal,  $\mathcal{K}_-$ , generators, form the  $su(1, 1)$  algebra:  $[\mathcal{J}^3, \mathcal{J}^\pm]_- = \pm \mathcal{J}^\pm$ ,  $[\mathcal{J}^+, \mathcal{J}^-]_- = -2\mathcal{J}^3$ . The vacuum state is  $\mathcal{O}_{jj}$ ,  $[\mathcal{J}^-, \mathcal{O}_{jj}]_- = 0$ . The eigenvalues of the Casimir operator  $\mathcal{J}^2 = \mathcal{J}^3(\mathcal{J}^3 - 1) - \mathcal{J}^+ \mathcal{J}^-$  define the conformal spin,  $J_{12}$ , of the state, i.e.  $[\mathcal{J}^2, \mathcal{O}_{jl}]_- = J_{12}(J_{12} + 1)\mathcal{O}_{jl}$  with  $J_{12} = j + \frac{1}{2}(\nu_1 + \nu_2)$ .

Therefore, the evolution equation can be reformulated into the eigenvalue problem for the three-particle system in a basis of local operators

$$\mathcal{H}_{\text{QCD}} \Psi = \mathcal{E}_{\text{QCD}} \Psi, \quad (19)$$

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<sup>3</sup>Note that from  $\mathcal{N} = 1$  SUSY Ward identities it follows that  ${}^{q\bar{q}}\gamma_j^T = {}^{qg}\gamma_j^T$  for  $j = \text{even}$ ,  ${}^{q\bar{q}}\gamma_{j+1}^T = {}^{qg}\gamma_j^T$  for  $j = \text{odd}$  at leading order and, therefore, as can be easily seen from the results displayed below in  $\mathcal{N} = 1$  super Yang-Mills theory the Hamiltonian of interaction of  $n$ -particles with the same helicity is equivalent to the one of the  $XXX_{s=-1}$  spin chain.

with Hamiltonian

$$\mathcal{H}_{\text{QCD}} = {}^{q\bar{q}}\mathcal{H}_{(8)}^T(\hat{J}_{13}) + {}^{qg}\mathcal{H}_{(3)}^V(\hat{J}_{12}) + {}^{qg}\mathcal{H}_{(3)}^V(\hat{J}_{32}) - \frac{\beta_0}{4}, \quad (20)$$

where we have defined  ${}^{q\bar{q}}\gamma_{J-1} = 2 {}^{q\bar{q}}\mathcal{H}(J)$  and  ${}^{qg}\gamma_{J-3/2} = 2 {}^{qg}\mathcal{H}(J)$ . The operators  $\hat{J}$ 's are formally determined as solutions of the equation  $\hat{\mathbf{J}}^2 = \hat{J}(\hat{J} + 1)$ .

In the large  $N_c$  limit Eq. (20) simplifies into

$$\mathcal{H} = \frac{N_c}{2} \left\{ \psi \left( \hat{J}_{12} - \frac{1}{2} \right) + \psi \left( \hat{J}_{12} + \frac{5}{2} \right) + \psi \left( \hat{J}_{32} - \frac{1}{2} \right) + \psi \left( \hat{J}_{32} + \frac{5}{2} \right) - 4\psi(1) - \frac{3}{2} \right\}. \quad (21)$$

Thus, the Hamiltonians (20,21) explicitly manifest the  $SU(1, 1)$  invariance of the system.

## 4 $\theta$ -space.

Let us explore in full the consequences of the covariance of the problem under the conformal transformations. For this we define a space  $V = \{\theta^k | k = 0, 1, \dots, \infty\}$  spanned by the elements<sup>4</sup>

$$\theta^k \equiv \frac{\partial_+^k \phi}{\Gamma(k + \nu + 1)}. \quad (22)$$

In the representation  $[\mathcal{J}^{\pm,3}, \chi(\theta)]_- = \hat{J}^{\pm,3} \chi(\theta)$  the generators are

$$\hat{J}^+ = (\nu + 1)\theta + \theta^2 \frac{\partial}{\partial \theta}, \quad \hat{J}^- = \frac{\partial}{\partial \theta}, \quad \hat{J}^3 = \frac{1}{2}(\nu + 1) + \theta \frac{\partial}{\partial \theta}, \quad (23)$$

with commutation relations:  $[\hat{J}^3, \hat{J}^\pm]_- = \pm \hat{J}^\pm$ ,  $[\hat{J}^+, \hat{J}^-]_- = -2\hat{J}^3$ . For a multi-variable function  $\chi(\theta_1, \theta_2, \dots, \theta_n)$  the operators are defined as  $\hat{J}^{\pm,3} = \sum_{\ell=1}^n \hat{J}_\ell^{\pm,3}$  and the quadratic Casimir operator is  $\hat{\mathbf{J}}^2 = \hat{J}^3(\hat{J}^3 - 1) - \hat{J}^+ \hat{J}^-$ .

Since the spectrum of eigenvalues of the problem (19) have to be real it means that the Hamiltonian has to be selfadjoint w.r.t. an appropriate scalar product. We define it as (see [15] for definition of group invariant measures)

$$\langle \chi(\theta_1, \theta_2, \dots, \theta_n) | \chi(\theta_1, \theta_2, \dots, \theta_n) \rangle = \int_{\Omega} d\mathcal{M} \chi(\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_n) \chi(\theta_1, \theta_2, \dots, \theta_n), \quad (24)$$

where

$$d\mathcal{M} \equiv \prod_{\ell=1}^n \frac{d\theta_\ell d\bar{\theta}_\ell}{2\pi i} (1 - \theta_\ell \bar{\theta}_\ell)^{\nu_\ell - 1} \quad \text{and} \quad \Omega = \bigcup_{\ell=1}^n \{|\theta_\ell| \leq 1\}$$

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<sup>4</sup>It is worth to note that we work in the space of local operators rather than with the correlation functions of composite operator,  $\mathcal{O}$ , with elementary fields,  $\phi_\ell$ ,  $\langle \mathcal{O} \prod_\ell \phi_\ell \rangle$ , or in the language of Refs. [14, 3] the so-called hat-transformed basis.

and  $\bar{\theta} = \theta^*$ . Then the following adjoint properties are obvious

$$\begin{aligned}\langle \chi(\theta_1, \theta_2, \dots, \theta_n) | \hat{J}^{\pm, 3} \chi(\theta_1, \theta_2, \dots, \theta_n) \rangle &= \langle \hat{J}^{\mp, 3} \chi(\theta_1, \theta_2, \dots, \theta_n) | \chi(\theta_1, \theta_2, \dots, \theta_n) \rangle, \\ \langle \chi(\theta_1, \theta_2, \dots, \theta_n) | \mathcal{H} \chi(\theta_1, \theta_2, \dots, \theta_n) \rangle &= \langle \mathcal{H} \chi(\theta_1, \theta_2, \dots, \theta_n) | \chi(\theta_1, \theta_2, \dots, \theta_n) \rangle.\end{aligned}$$

Now we are ready to address the question of construction of irreducible representations in  $\theta$ -space.

## 5 Two-point basis.

The highest weight vector in the  $\theta$ -space depending on two variables,  $\hat{J}_{12}^- \chi(\theta_1, \theta_2) = 0$ , is realized by translation invariant polynomials<sup>5</sup>  $\chi(\theta_1, \theta_2) = \theta_{12}^j$ . The descendants are constructed by acting with the step-up operator on the vacuum state:  $(\hat{J}_{12}^+)^k \theta_{12}^j$ . When transformed to the basis of the local operators (22) they coincide with Eq. (18) up to an overall normalization

$$\mathcal{O}_{j,j+k} = i^{j+k} \frac{\Gamma(j + \nu_1 + 1) \Gamma(j + \nu_2 + 1)}{\Gamma(j + 1)} (\hat{J}_{12}^+)^k \theta_{12}^j.$$

The states with unit norm w.r.t. the scalar product (24) are

$$\mathcal{P}_j(\theta_1, \theta_2) = n^{-1}(j|\nu_1, \nu_2) \theta_{12}^j, \quad \text{with} \quad n^2(j|\nu_1, \nu_2) = \frac{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(j + 1) \Gamma(2j + \nu_1 + \nu_2 + 1)}{\Gamma(j + \nu_1 + 1) \Gamma(j + \nu_2 + 1) \Gamma(j + \nu_1 + \nu_2 + 1)}, \quad (25)$$

so that  $\langle \mathcal{P}_{j'}(\theta_1, \theta_2) | \mathcal{P}_j(\theta_1, \theta_2) \rangle = \delta_{j'j}$ . The two-particle Casimir operator

$$\hat{\mathbf{J}}_{12}^2 = -\theta_{12}^2 \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} - \theta_{12} \left( (\nu_1 + 1) \frac{\partial}{\partial \theta_2} - (\nu_2 + 1) \frac{\partial}{\partial \theta_1} \right) + \frac{\nu_1 + \nu_2}{2} \left( \frac{\nu_1 + \nu_2}{2} + 1 \right) \quad (26)$$

is obviously diagonal in the basis

$$\langle \mathcal{P}_{j'}(\theta_1, \theta_2) | \hat{\mathbf{J}}_{12}^2 | \mathcal{P}_j(\theta_1, \theta_2) \rangle = \delta_{j'j} [\hat{\mathbf{J}}_{12}^2]_{jj}, \quad \text{with} \quad [\hat{\mathbf{J}}_{12}^2]_{jj} = \left( j + \frac{\nu_1 + \nu_2}{2} \right) \left( j + \frac{\nu_1 + \nu_2}{2} + 1 \right). \quad (27)$$

## 6 Three-point basis.

In order to construct an orthonormal basis of three-particles operators we expand them w.r.t. the eigenfunctions of the two-point Casimir operator, say  $\hat{\mathbf{J}}_{12}^2$ , as follows

$$\mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) = \sum_{k=0}^{J-j} P_k \theta_3^{J-j-k} (\hat{J}_{12}^+)^k \mathcal{P}_j(\theta_1, \theta_2). \quad (28)$$

---

<sup>5</sup>We use everywhere the shorthand notation  $\theta_{\ell\ell'} \equiv \theta_\ell - \theta_{\ell'}$ .



From the condition<sup>6</sup>  $\hat{J}^- \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) = 0$  we deduce the expansion coefficients

$$P_k = N^{-1}(J, j | \nu_1, \nu_2, \nu_3) \frac{(-1)^k \Gamma(J - j + 1)}{\Gamma(k + 1) \Gamma(J - j - k + 1)} \frac{\Gamma(2j + \nu_1 + \nu_2 + 2)}{\Gamma(2j + k + \nu_1 + \nu_2 + 2)}, \quad (29)$$

where the factor

$$N^2(J, j | \nu_1, \nu_2, \nu_3) = \frac{\Gamma(\nu_3) \Gamma(J - j + 1) \Gamma(2j + \nu_1 + \nu_2 + 2) \Gamma(2J + \nu_1 + \nu_2 + \nu_3 + 2)}{\Gamma(J - j + \nu_3 + 1) \Gamma(J + j + \nu_1 + \nu_2 + 2) \Gamma(J + j + \nu_1 + \nu_2 + \nu_3 + 2)}. \quad (30)$$

ensures the normalization of the state to unity  $\langle \mathcal{P}_{J';j'}(\theta_1, \theta_2 | \theta_3) | \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) \rangle = \delta_{j'j} \delta_{J'J}$ . From the construction it is obvious that  $\hat{\mathbf{J}}_{12}^2$  and total Casimir operator

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_{12}^2 + \hat{\mathbf{J}}_{23}^2 + \hat{\mathbf{J}}_{13}^2 + \sum_{\ell=1}^3 \frac{1 - \nu_\ell^2}{4}, \quad (31)$$

which is the sum of the two-particle Casimir operators in subchannels minus the single-particle ones, are diagonal in the basis  $\mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3)$

$$\langle \mathcal{P}_{J';j'}(\theta_1, \theta_2 | \theta_3) | \hat{\mathbf{J}}_{12}^2 | \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) \rangle = \delta_{J'J} \delta_{j'j} [\hat{\mathbf{J}}_{12}^2]_{jj}, \quad (32)$$

$$\langle \mathcal{P}_{J';j'}(\theta_1, \theta_2 | \theta_3) | \hat{\mathbf{J}}^2 | \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) \rangle = \delta_{J'J} \delta_{j'j} [\hat{\mathbf{J}}^2]_{jj}, \quad (33)$$

where  $[\hat{\mathbf{J}}^2]_{jj} = [\hat{J}_3]_{jj} ([\hat{J}_3]_{jj} - 1)$  and  $[\hat{J}_3]_{jj} = J + \frac{1}{2}(\nu_1 + \nu_2 + \nu_3 + 3)$ . Since the main quantum number  $J$  (total conformal spin) is conserved we do not display the dependence on it in matrix elements.

The matrix elements of the remaining generators can be easily evaluated and the result is

$$\langle \mathcal{P}_{J';j'}(\theta_1, \theta_2 | \theta_3) | \hat{\mathbf{J}}_{23}^2 | \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) \rangle = \delta_{J'J} \left( \delta_{j'j} [\hat{\mathbf{J}}_{23}^2]_{jj} + \delta_{j',j+1} [\hat{\mathbf{J}}_{23}^2]_{j+1,j} + \delta_{j',j-1} [\hat{\mathbf{J}}_{23}^2]_{j-1,j} \right) \quad (34)$$

where the diagonal part is

$$[\hat{\mathbf{J}}_{23}^2]_{jj} = \frac{1}{2} \left( [\hat{\mathbf{J}}^2]_{jj} - [\hat{\mathbf{J}}_{12}^2]_{jj} - \sum_{\ell=1}^3 \frac{1 - \nu_\ell^2}{4} + \frac{\nu_2^2 - \nu_1^2}{4} \frac{1}{[\hat{\mathbf{J}}_{12}^2]_{jj}} \left( [\hat{\mathbf{J}}^2]_{jj} + \frac{1 - \nu_3^2}{4} \right) \right), \quad (35)$$

and the non-diagonal elements are

$$[\hat{\mathbf{J}}_{23}^2]_{j+1,j} = \frac{\mathcal{N}(J, j)}{\mathcal{N}(J, j+1)} (j+1)(J-j+\nu_3), \quad [\hat{\mathbf{J}}_{23}^2]_{j-1,j} = \frac{\mathcal{N}(J, j-1)}{\mathcal{N}(J, j)} j(J-j+\nu_3+1), \quad (36)$$

with  $\mathcal{N}(J, j) \equiv N(J, j | \nu_1, \nu_2, \nu_3) n(j | \nu_1, \nu_2)$ . So that  $[\hat{\mathbf{J}}_{23}^2]_{j+1,j} = [\hat{\mathbf{J}}_{23}^2]_{j,j+1}$ .

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<sup>6</sup>In what follows operators without subscripts stand for three-particle ones while two-particle operators in subchannels are labeled by corresponding indices.

## 7 Integral of motion and its quantization.

On top of the conformal invariance there exists another hidden symmetry of the system (21). The beautiful finding made by the authors of Ref. [3] is the identification of an additional conserved charge of the three-particle problem<sup>7</sup> described by the Hamiltonian (21) in the limit of large number of colours,

$$\mathcal{Q}_T = [\hat{\mathbf{J}}_{12}^2, \hat{\mathbf{J}}_{23}^2]_+ - \frac{9}{2} \left\{ \hat{\mathbf{J}}_{12}^2 + \hat{\mathbf{J}}_{23}^2 \right\}, \quad (37)$$

which is the hermitian operator and commutes with the Hamiltonian (21)  $[\mathcal{H}, \mathcal{Q}_T]_- = 0$ . Everywhere we have to put  $\nu_1 = \nu_3 = 1$ ,  $\nu_2 = 2$ .

The existence of this additional charge leads to complete integrability of the system (the number of integrals of motion equals the number of degrees of freedom). This allows to reduce the complicated eigenfunction problem for the Hamiltonian (21) to the more simple one for the  $\mathcal{Q}_T$ :

$$\mathcal{Q}_T \Psi = q_T \Psi. \quad (38)$$

We will look for the solution in the form of expansion w.r.t. the three-point basis (28), i.e.

$$\Psi = \sum_{j=0}^J \Psi_j \mathcal{P}_{J;j}. \quad (39)$$

Its main advantage comes from the fact that the matrix elements of the charge possess only three non-zero diagonals:

$$\langle \mathcal{P}_{J';j'}(\theta_1, \theta_2 | \theta_3) | \mathcal{Q}_T | \mathcal{P}_{J;j}(\theta_1, \theta_2 | \theta_3) \rangle = \delta_{J'J} (\delta_{j'j} [\mathcal{Q}_T]_{jj} + \delta_{j',j+1} [\mathcal{Q}_T]_{j+1,j} + \delta_{j',j-1} [\mathcal{Q}_T]_{j-1,j}), \quad (40)$$

with

$$[\mathcal{Q}_T]_{jj} = 2 [\hat{\mathbf{J}}_{12}^2]_{jj} [\hat{\mathbf{J}}_{23}^2]_{jj} - \frac{9}{2} \left\{ [\hat{\mathbf{J}}_{12}^2]_{jj} + [\hat{\mathbf{J}}_{23}^2]_{jj} \right\}, \quad (41)$$

$$[\mathcal{Q}_T]_{j+1,j} = \left\{ [\hat{\mathbf{J}}_{12}^2]_{j+1,j+1} + [\hat{\mathbf{J}}_{12}^2]_{jj} - \frac{9}{2} \right\} [\hat{\mathbf{J}}_{23}^2]_{j+1,j}, \quad (42)$$

and  $[\mathcal{Q}_T]_{j+1,j} = [\mathcal{Q}_T]_{j,j+1}$ . Therefore, the above equation (38) leads to a three-term recursion relation for  $\Psi_j$  which when solved gives the expansion coefficients as well as quantized values of  $\mathcal{Q}_T$ . The former is of the form

$$([\mathcal{Q}_T]_{j,j} - q_T) \Psi_j + [\mathcal{Q}_T]_{j,j-1} \Psi_{j-1} + [\mathcal{Q}_T]_{j,j+1} \Psi_{j+1} = 0. \quad (43)$$

with the boundary conditions  $\Psi_{-1} = \Psi_{J+1} = 0$ . The solution for this recursion relation which satisfies the boundary conditions exists provided the following constraint is fulfilled

$$([\mathcal{Q}_T]_{j,j} - q_T)^2 \leq 4 [\mathcal{Q}_T]_{j,j+1} [\mathcal{Q}_T]_{j,j-1}. \quad (44)$$

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<sup>7</sup>See Ref. [16] for an exhaustive treatment of the three-quark problem in the context of solution of the Brodsky-Lepage evolution equation for baryon distribution amplitudes.

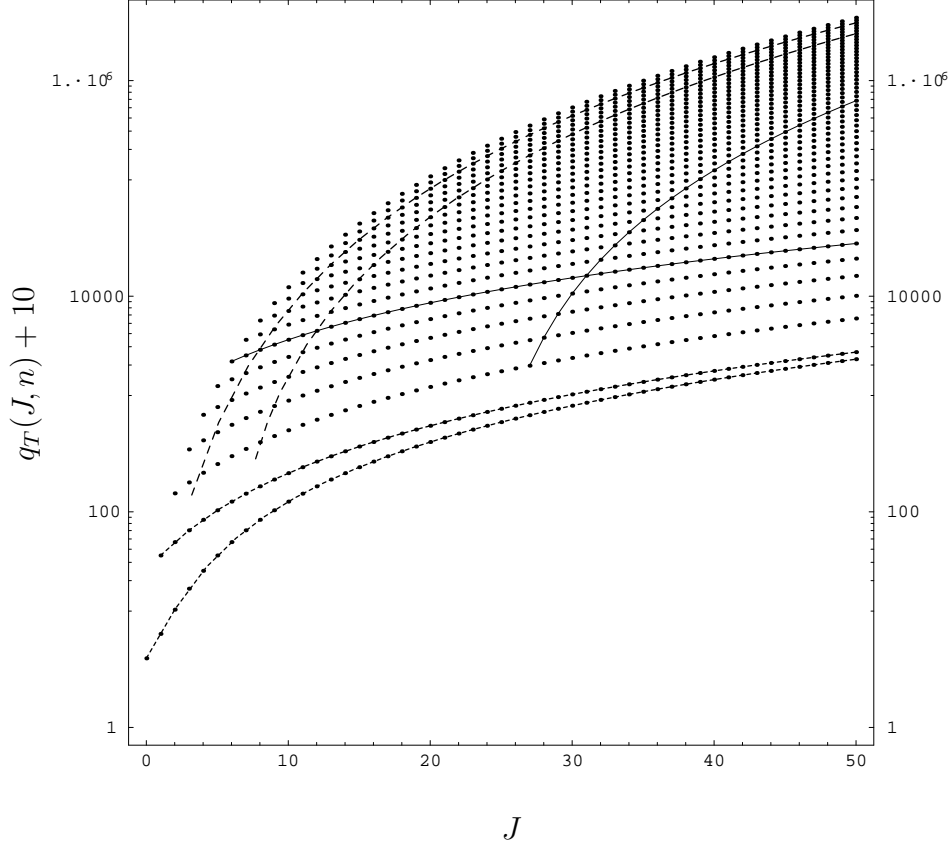


Figure 3: The spectrum of the conserved charge  $q_T$ . Sample trajectories from two sets: for  $n = 25$  (counts from above,  $n = 0, 1, \dots$ ) and  $m = 7$  (counts from below,  $m = 1, 2, \dots$ ) are shown by solid curves. Long-dashed lines correspond to the analytical formulae (46,50) with  $n = 2, 6$ . The two exact solutions (51) separated from the rest of the spectrum by a finite gap are demonstrated by short-dashed lines.

Using the results found so far we can easily deduce from the Eq. (44) the critical eigenvalues of the spectrum of the operator  $\mathcal{Q}_T$  for large conformal spin  $J$  of the three particle system. Substituting the asymptotics for the matrix element of the operator  $\mathcal{Q}_T$  in the three-point basis  $[\mathcal{Q}_T]_{j,j}|_{J \rightarrow \infty} = 2[\mathcal{Q}_T]_{j,j+1}|_{J \rightarrow \infty} = J^4 \tau^2 (1 - \tau)^2$  where  $\tau \equiv j/J$ , in Eq. (44) we get  $0 \leq q_T \leq 2J^4 \tau^2 (1 - \tau^2)$ . The maximum is achieved for  $\tau_{\max} = 1/\sqrt{2}$ . Thus the recurrence relation possesses the rising solution for  $0 \leq j \ll \frac{1}{\sqrt{2}}J$  and decreasing one for  $\frac{1}{\sqrt{2}}J \ll j \leq J$ . We get finally an estimate for the quantized values of the  $\mathcal{Q}_T$  for  $J \rightarrow \infty$ . Namely, the eigenvalues lie in the strip

$$0 \leq q_T/J^4 \leq \frac{1}{2}. \quad (45)$$

Note that numerically the upper bound on the spectrum is attained at very high  $J$ 's. For instance, for  $J = 10^3$  we have  $1.00199 \cdot 10^{-6} \leq q_T/J^4 \leq 0.50531$ .

We are now in a position to estimate non-leading in  $1/J^\ell$  corrections which will generate the fine

structure of the spectrum. Let us consider the upper bound of the spectrum. Since the maximal eigenvalues are achieved for  $j_{\max} = \frac{1}{\sqrt{2}}J$  we consider a small vicinity of  $j_{\max}$ :  $j = \frac{1}{\sqrt{2}}(J + \lambda\sqrt{J})$ . We look for the expansion of the eigenfunctions  $\Psi_j \equiv \Phi(\lambda)$  and the charge  $q_T$  in series

$$\Phi(\lambda) = \sum_{\ell=0}^{\infty} \Phi_{(\ell)}(\lambda) J^{-\ell/2}, \quad \text{and} \quad q_T(J, n) = J^4 \sum_{\ell=0}^{\infty} q_T^{(\ell)}(n) J^{-\ell}, \quad \text{with} \quad q_T^{(0)} = \frac{1}{2}, \quad (46)$$

respectively. Then the difference equation (43) is replaced by the sequence of coupled differential equations

$$\Phi_{(0)}''(\lambda) + 4(6 - q_T^{(1)} - 2\lambda^2) \Phi_{(0)}(\lambda) = 0, \quad (47)$$

$$\Phi_{(1)}''(\lambda) + 4(6 - q_T^{(1)} - 2\lambda^2) \Phi_{(1)}(\lambda) = 8\lambda(4\sqrt{2} - 6 + \lambda^2) \Phi_{(0)}(\lambda), \quad (48)$$

...

The solution of the first one, which satisfies the boundary conditions, i.e.  $\Phi(\pm\infty) = 0$ , is expressed by Hermite polynomials

$$\Phi_{(0)}(\lambda) = H_n \left( \sqrt{2\sqrt{2}\lambda} \right) e^{-\sqrt{2}\lambda^2}, \quad (49)$$

giving the quantized values of  $q_T^{(1)}$

$$q_T^{(1)}(n) = \sqrt{2} \left( 3\sqrt{2} - \frac{1}{2} - n \right), \quad n = 0, 1, \dots \quad (50)$$

Here for a given  $J$ ,  $0 \leq n \leq J$  but the Eq. (47) has been derived in the approximation  $n \ll J$  and thus we can describe the upper part of the spectrum only. The comparison of this approximation with the eigenvalues evaluated numerically is shown in Fig. 3. The agreement is reasonable already with the first non-leading correction taken into account. We can go to the region  $n \sim J$  provided a large enough number of terms is kept in Eqs. (46).

While for low part of the spectrum one can find two exact solutions:

$$q_T^{\text{exact}-1}(J) = -\frac{53}{8} + (J+1)^2, \quad q_T^{\text{exact}-2}(J) = -\frac{53}{8} + (J+5)^2, \quad (51)$$

separated from the rest of the spectrum by a finite gap of order  $\Delta q_T(J) \propto 0.7J^2 + 40J$  for large  $J$ 's.

## 8 Eigen-energy of three-particle system.

The eigenfunctions of the integral of motion  $\mathcal{Q}_T$  simultaneously diagonalize the Hamiltonian (21) and, therefore, the energy of the system is<sup>8</sup>

$$\mathcal{E}(J, n) = \frac{N_c}{2} \left\{ 2 \left( \sum_{j=0}^J |\Psi_j|^2 \right)^{-1} \sum_{j=0}^J \epsilon(j) |\Psi_j|^2 - \frac{3}{2} \right\}, \quad (52)$$

---

<sup>8</sup>We have used the permutation symmetry of the quark fields so that  $P_{13}\Psi = e^{i\varphi}\Psi$ , and  $\sum_j \Psi_j \mathcal{P}_{J;j}(\theta_3, \theta_2|\theta_1) = e^{i\varphi} \sum_j \Psi_j \mathcal{P}_{J;j}(\theta_1, \theta_2|\theta_3)$  since  $P_{13}\mathcal{P}_{J;j}(\theta_1, \theta_2|\theta_3) = e^{i\varphi} \mathcal{P}_{J;j}(\theta_3, \theta_2|\theta_1)$  where  $\varphi = 0, \pi$ .

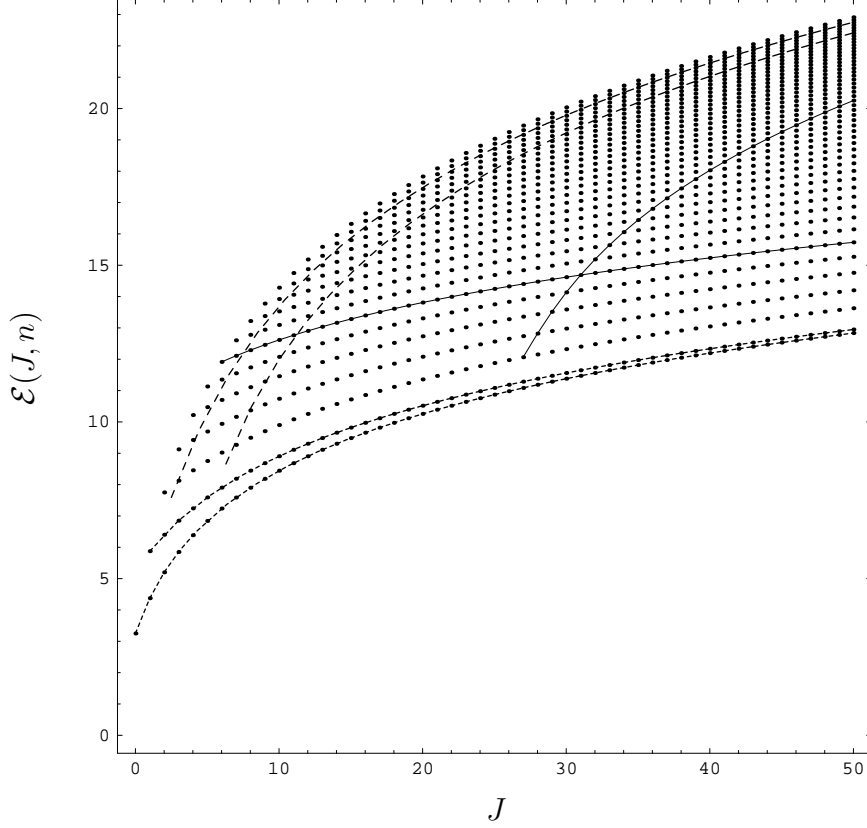


Figure 4: Same as in Fig. 3 but for the spectrum of energy eigenstates. The two lowest exact trajectories are calculated from Eqs. (58).

with  $\epsilon(j) = \psi(j+1) + \psi(j+4) + 2\gamma_E$ . The upper limit on the spectrum can easily be read from the fact that the function  $\Psi_j$  is peaked at  $j_{\max}$  and thus  $\mathcal{E}^{\max}(J) = N_c/2 \left\{ 2\epsilon \left( J/\sqrt{2} \right) - \frac{3}{2} \right\}$ . But this estimate contains non-leading terms as well.

The energy can be evaluated consistently in WKB approximation as

$$\mathcal{E}(J, n) = \frac{N_c}{2} \left\{ \mathcal{E}^{(0)}(J) + \sum_{\ell=1}^{\infty} \mathcal{E}^{(\ell)}(n) J^{-\ell} \right\}. \quad (53)$$

Combining Eqs. (53) and (52) with known eigenfunctions allow to determine the coefficients  $\mathcal{E}^{(\ell)}$ . However, to evaluate next-to-leading WKB correction  $\mathcal{E}^{(1)}$  we have to find  $\Phi_{(1)}$  from Eq. (48) as well. The solution of this equation is given by

$$\Phi_{(1)}(\lambda) = \int_{-\infty}^{\infty} d\lambda' G(\lambda, \lambda') \mathcal{J}(\lambda'). \quad (54)$$

Here the source  $\mathcal{J}(\lambda) = 8\lambda \left( 4\sqrt{2} - 6 + \lambda^2 \right) \Phi_{(0)}(\lambda)$  and the Green function of the homogeneous equation is

$$G(\lambda, \lambda') = \frac{1}{2\sqrt{2}\sqrt{2}} \sum_{m=0}^{\infty} \frac{e^{-\sqrt{2}(\lambda^2 + \lambda'^2)}}{\sqrt{\pi} 2^m m! (n-m)} H_m \left( \sqrt{2}\sqrt{2}\lambda \right) H_m \left( \sqrt{2}\sqrt{2}\lambda' \right). \quad (55)$$

We have omitted, however, in Eq. (54) the solution of the homogeneous equation which might be added due to presence of a zero-mode (for  $m = n$ ), since in the energy the corresponding contribution disappears being an odd function of  $\lambda$ . Substituting finally the expansion (46) in Eq. (52) approximating the sum by the integral in the vicinity of the maximum of  $\Psi_j$  we find the energy to  $\mathcal{O}(J^{-1})$  accuracy  $\mathcal{E} = N_c \{ \epsilon^{(0)} - 3/4 + J^{-1} (2 \int \epsilon^{(1)} \Phi_{(0)} \Phi_{(1)} + \int \epsilon^{(2)} \Phi_{(0)}^2) / (\int \Phi_{(0)}^2) \}$  with  $\epsilon^{(0)} = 2 \ln J - \ln 2 + 2\gamma_E$ ,  $\epsilon^{(1)} = 2\lambda$ ,  $\epsilon^{(2)} = 4\sqrt{2} - \lambda^2$ . This gives us finally

$$\mathcal{E}^{(0)}(J) = 4 \ln J + 4\gamma_E - 2 \ln 2 - \frac{3}{2}, \quad \mathcal{E}^{(1)}(n) = 2\sqrt{2} \left( 3\sqrt{2} - \frac{1}{2} - n \right). \quad (56)$$

The restriction to this approximation for the energy works somewhat worse then for the  $q_T$ -charge (see Fig. 4) but can be improved routinely. From these results we derive the following expression for the energy via the integral of motion at large conformal spin

$$\mathcal{E} = \ln(q_T/2) + \text{const} + \mathcal{O}(J^{-2}), \quad (57)$$

which should be compared with  $\mathcal{E}_{XXX} = 2 \ln q + \text{const} + \mathcal{O}(J^{-2})$  for the closed  $XXX$  quantum spin chain [17].

The lowest trajectories corresponding to the solutions (51) reads

$$\begin{aligned} \mathcal{E}^{\text{exact}-1}(J) &= \frac{N_c}{2} \left\{ 2\psi(J+3) + 2\gamma_E - \frac{1}{2} - \frac{1}{J+3} \right\}, \\ \mathcal{E}^{\text{exact}-2}(J) &= \frac{N_c}{2} \left\{ 2\psi(J+3) + 2\gamma_E - \frac{1}{2} + \frac{3}{J+3} \right\}. \end{aligned} \quad (58)$$

And these are exactly the anomalous dimensions found in Ref. [18] (see also [19, 8]).

## 9 Outlook and conclusions.

To conclude we have constructed a convenient orthonormal three-particle basis in the space of local composite operators. Making use of it we have found that the allowed eigenvalues of the “hidden” integral of motion  $\mathcal{Q}_T$  and the energy  $\mathcal{E}$  of the system lie in the strips for large  $J$ ’s:  $J^2 \leq q_T \leq \frac{1}{2}J^4$  and  $2 \ln J \leq \mathcal{E} \leq 4 \ln J$  with the two lowest levels given by Eqs. (51,58) separated by finite gaps from the rest of the spectrum. The first non-leading WKB corrections which describe the fine structure of the spectra have been found. An improvement of these approximations could systematically be achieved by taking into account further  $\mathcal{O}(J^{-\ell})$  ( $\ell \geq 2$ ) terms in the recursion relation (43). However, this procedure becomes non-trivial for higher WKB corrections. Another fruitful way for determination the energy of the system would be the identification of the corresponding three-site integrable open spin chain model [20] with the charge (37) defined from the auxiliary transfer matrix  $\hat{t}(\lambda) = \text{Tr} (T(\lambda)K^-(\lambda)T^{-1}(-\lambda)K^+(\lambda))$  with boundary operators  $K^\pm$

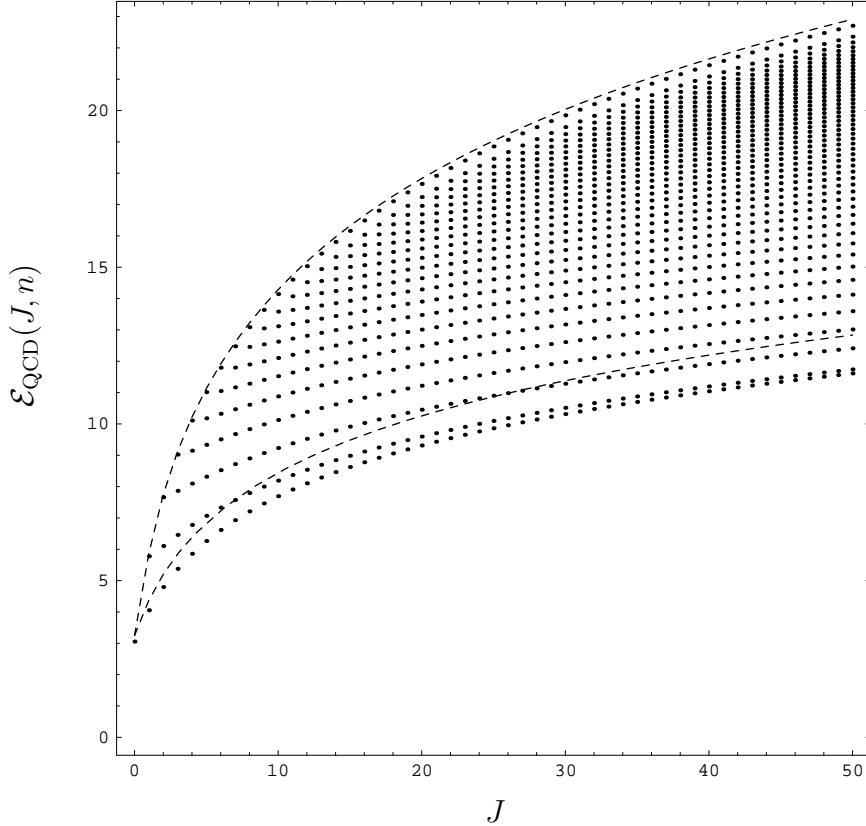


Figure 5: The spectrum of eigenvalues of Hamiltonian (20) compared to the range of eigenvalues in the multicolour limit (dashed curves).

satisfying the Sklyanin equation and auxiliary monodromy matrix  $T(\lambda) = \prod_{\ell=1}^3 L_{\ell}(\lambda)$ , given by the product of Lax operators  $L(\lambda) = \lambda \mathbb{1} + \sigma_+ \hat{J}^- - \sigma_- \hat{J}^+ + \sigma_3 \hat{J}^3$ , which fulfills the ordinary Yang-Baxter equation. However, since the Hamiltonian  $\mathcal{H}$  (21) of the system is essentially quantum and does not enter into the expansion of  $\hat{t}(\lambda)$  in rapidity  $\lambda$ , one has to construct the fundamental transfer matrix as well. Once the functional Bethe ansatz is developed it would allow to construct the WKB solution [17, 21] of the corresponding Baxter equation and as a result to deduce higher WKB corrections to the  $q_T$  in a more economic way together with the corresponding energy. Unfortunately the theory of open quantum spin chains [22] is far from being developed to the level of periodic models.

Of course, for real QCD case (finite  $N_c$ ) the integrability of the system is violated by  $1/N_c^2$  corrections in Eq. (20). These effects manifest themselves in the phenomenon of generation of a mass gap between the  $n = 0$  trajectory and the rest of the spectrum (see Fig. 5).

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